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Measuring the Accuracy of an Ancient Area Formula

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Abstract

In the ancient world, geometers were concerned primarily with mensuration (the practice of accurate measurement), with the most obvious applications being in construction and surveying. One well-known formula from this time (appearing most famously in the Temple of Horus in Egypt, c. 237-57 BCE) purports to give the area of a general quadrilateral by averaging the lengths of opposite pairs of sides and then multiplying the averages. While this formula is erroneous, it produces highly accurate results when the quadrilateral is nearly rectangular. We examine the relative accuracy of this area formula, including: (1) different methods of finding the exact area, (2) how to find the interior angle that minimizes the error of the formula, and (3) how significantly the error varies as the interior angle varies from the ideal. In the end, these observations lead to an improved version of the formula that relies only on the side-lengths of a quadrilateral.

1 Introduction

Imagine yourself living in ancient Egypt, about 2,000 years ago. You are a surveyor, and one of your tasks is to measure the size of the various farm fields in the area, so that the local authorities can tax the farmers. All of the fields under your jurisdiction are quadrilaterals; many are very nearly rectangular, but some of them have a skewed shape. How do you complete this task? Also, assuming your measurements are accurate, what data do you need to find the exact area of a field?

This scenario is an example of *mensuration*: the practice of accurate measurement. Over time, builders and surveyors in the ancient world developed a collection of mathematical formulas to expedite the calculation process. These formulas were not justified by rigorous proof, but rather by experience. If a formula provided a measure of area or volume that was indistinguishable from the true value, then it was considered correct.

First, let us consider some of the background on the problem. Since the sources here are scarce, archaeologists and historians have based their conclusions on a handful of well-known sources. One novel source of information is the Temple of Horus in Edfu, Egypt, whose inscriptions were first published in the West by Lepsius [5, p. 75 ff.] in 1855.[1] At the dedication of the temple, several tracts of land were dedicated to Horus and donated to the temple; both the dimensions and area of each field are given in the temple inscriptions. As recorded by Thomas Heath in his landmark text *A History of Greek Mathematics* [2, p. 124],

[1] An impressive collection of information on the temple has been complied and made available online by the Edfu Project: `http://www.edfu-projekt.gwdg.de/`

> From so much of these inscriptions as were published by Lepsius we gather that $\frac{1}{2}(a+c)\cdot\frac{1}{2}(b+d)$ was a formula for the area of a quadrilateral the sides of which are in order a, b, c, d.

In other words: average the lengths of opposite pairs of sides and then multiply the averages. (We call this formula the *Surveyor's Formula*, in keeping with Gupta [1].) This formula appears in many different cultures over the course of many centuries, so it is not unique to the Egyptians (see Gupta [1] for a comprehensive listing of appearances of the Surveyor's Formula).

In a sense, the Surveyor's Formula "forces" the quadrilateral to be a rectangle with side-lengths of $\frac{a+c}{2}$ and $\frac{b+d}{2}$, and then calculates the area of the rectangle (see Figure 1).

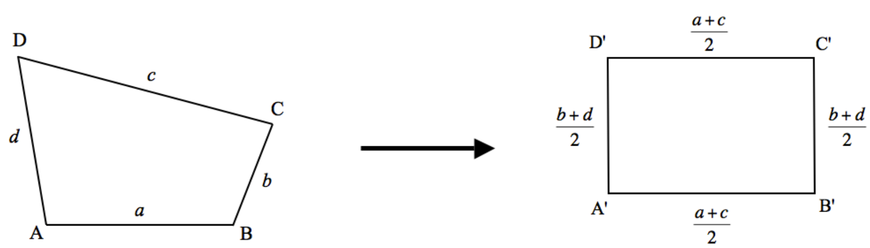

Figure 1. The "forcing" of a quadrilateral into a rectangular shape.

However, it is easy to see that this formula is incorrect. For example, choose any nonrectangular parallelogram and you will find the formula overestimates its area. This observation is nothing new: as Heath noted, "It is remarkable enough that the use of a formula so inaccurate should have lasted till 200 years after Euclid had lived and taught in Egypt" [2, p. 124]. Furthermore, it is known that this formula will never underestimate the true area of the quadrilateral (see Gupta [1, p. 55] or Pottage [6, p. 302] for a proof of this fact).

But let us return to your task as an ancient Egyptian surveyor. Your goal is to use a formula that is both accurate (as far as you are able to tell) and effective (it is quick and easy to use) for the farm fields under your jurisdiction. Our goal in this article is to assess the relative accuracy of one such formula—the Surveyor's Formula. After reviewing the various formulae for calculating the *exact* area of a quadrilateral, we use numerical methods to determine the relative error of Surveyor's Formula. Then, we use these methods to analyze some specific examples, two of which are themselves taken from the Temple of Horus. Lastly, we generate some more comprehensive data that point toward a more accurate formula.

2 Orienting the Quadrilateral

As a surveyor, your task is relatively simple: first choose a corner of the field, and then walk around the perimeter while making note of the side-lengths. Geometrically, this is equivalent to choosing a corner (one of A, B, C, or D) and an *orientation* (clockwise or counterclockwise). With this in mind, a generic quadrilateral $ABCD$ will be labeled counterclockwise so that AB (taken to be the base) has length a, BC has length b, CD has length c, and DA has length d; this is implicit in Figure 1.

For you, this information is quite enough: once the side-lengths are known, you can apply the averaging technique of the Surveyor's Formula and report the result to your supervisors. However, the side-lengths alone are not sufficient to produce a well-defined figure: one may change the interior angles of a quadrilateral while leaving the side-lengths fixed. Thus, any *exact* calculation of area requires at least one more piece of information. Typically, there are two ways to do this: the first is to take the measure of one interior angle, while the other is to take the average of a pair of opposite interior angles.

While we will decline to use the angle-averaging method in the end, it will aid in our analysis of the single-area formula, so let us first examine the formula that it produces. To do this, first relabel the angles $\angle B$ and $\angle D$ as θ and φ, respectively.

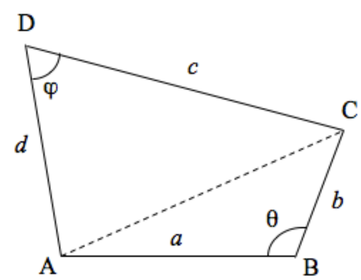

Figure 2. A quadrilateral with all vertices and sides labeled, along with two opposite angles.

Letting $\psi = \frac{1}{2}(\theta + \varphi)$, the formula for the exact area is

$$\text{Area} \;=\; \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\psi}\,, \tag{1}$$

where $s = \frac{1}{2}(a+b+c+d)$ is the semiperimeter (see Gupta [1, p. 53]). It's easy to see that the area is maximized when ψ is a right angle, so that $\cos\psi = 0$. In this case, the formula is identical to Brahmagupta's formula (see Kichenassamy [3, p. 29] or Kusuba [4, p. 52]) for the area of a cyclic quadrilateral.

While this formula is elegant, and its maximum value is easily calculated, it is not the most practical way to calculate the area. From your perspective as a surveyor, it would be simpler to take the measure of a single interior angle. However, even this task is ambiguous: any one of the four interior angles could give rise to a valid formula. To eliminate this ambiguity, we need to establish an orientation to be used for all quadrilaterals we encounter.

Definition 1. *For any quadrilateral, we label the vertices A, B, C, D in a counterclockwise fashion (also labeling the corresponding side-lengths as $AB = a$, $BC = b$, $CD = c$,*

$DA = d$) so that $a+b \le c+d$ and $a+d \le b+c$. We will refer to this as the standard orientation *of a quadrilateral.*

This orientation is easy to accomplish. The two inequalities can be written as $c-a \ge b-d$ and $c-a \ge d-b$, so we may choose the pair of opposite sides with the greater difference between them and label them as a and c with $c \ge a$.

3 The Formula for Exact Area

The simplest way to calculate the exact area of a quadrilateral (using the standard orientation with additional labeling as given in Figure 2) is to slice it along the diagonal AC and then add the areas of the resultant triangles:

$$\text{Area} \;=\; \frac{1}{2}ab\sin\theta \,+\, \frac{1}{2}cd\sin\varphi. \tag{2}$$

Of course, this depends on two interior angles instead of one. We eliminate φ with the following result.

Theorem 1. *Given a quadrilateral $ABCD$ with the standard orientation, its exact area may be obtained from the formula*

$$A(\theta) \;=\; \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sqrt{1-(L+M\cos\theta)^2}, \tag{3}$$

where $L = \frac{c^2+d^2-a^2-b^2}{2cd}$, $M = \frac{ab}{cd}$, and θ is the measure of the angle $\angle B$.

The standard orientation is necessary in order to assign unambiguous labels to the four sides and the angle θ. The truth of the theorem follows readily from the law of cosines, since the two triangles ABC and ACD share the common side AC. Thus, we may use the relation between $\cos\theta$ and $\cos\varphi$ to express $\sin\varphi$ in terms of $\cos\theta$. Next, we want to know how accurate the Surveyor's Formula will be in practice. Treating the side-lengths as fixed, it is easy to find the minimum possible error: simply maximize the function $A(\theta)$.

Theorem 2. *The single-variable formula $A(\theta)$ is maximized when $\theta = \theta_0 = \arccos(\frac{-L}{M+1})$.*

This result follows from the fact that the angle-averaging area formula (1) is maximized when $\varphi + \theta = \pi$, which provides the relation $\cos\varphi = -\cos\theta$. Then, taking θ_0 and φ_0 to be the angles at which the area is maximized, the proof of the single-angle formula (3) tells us that

$$-\cos\theta_0 \;=\; \cos\varphi_0 \;=\; L + M\cos\theta_0, \tag{4}$$

from which we obtain $\cos\theta_0 = \frac{-L}{M+1}$.

4 Measures of Average

Since formula (3) treats area as a function of the single, continuous variable θ, integration is the simplest way to calculate the average area over any interval $[\theta_1, \theta_2]$:

$$\text{Average area over } [\theta_1, \theta_2] \quad = \quad \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} A(\theta)\, d\theta. \tag{5}$$

We also define the relative error as a function of θ:

$$R(\theta) \;=\; \frac{|E - A(\theta)|}{A(\theta)} \;=\; \left| \frac{E}{A(\theta)} - 1 \right|,$$

where E denotes the estimated area given by the Surveyor's Formula. (Since the Surveyor's Formula depends only on the side-lengths, it follows that E is constant with regard to θ.) Lastly, the average relative error for an interval $[\theta_1, \theta_2]$ is obtained by integrating $R(\theta)$:

$$\frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} R(\theta)\, d\theta \;=\; \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \left| \frac{E}{A(\theta)} - 1 \right| d\theta. \tag{6}$$

In order to apply these formulas, we need to find the minimum and maximum values of θ for a given set of side-lengths. This is where the standard orientation will help us: since $b + c \geq a + d$, minimizing θ will produce a triangle with side-lengths $a + d$, b, and c. Again using the law of cosines, we find that $\cos\theta = \frac{(a+d)^2+b^2-c^2}{2(a+d)b}$ in this case (let θ_1 denote this angle). Since $c + d \geq a + b$, maximizing θ will produce a triangle with sides $a + b$, c, and d, and $\theta_2 = \pi$. With these bounds, we can now apply formulas (5) and (6) to obtain the average area and average relative error over all possible θ values.

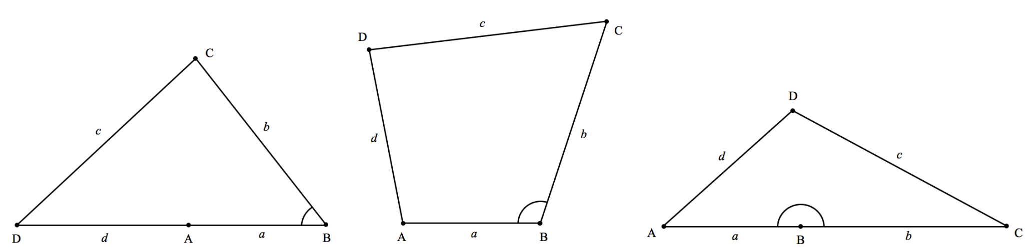

Figure 3. Finding the minimum (left) and maximum (right) θ values for a quadrilateral $ABCD$ (center).

Example 1: $(a, b, c, d) = (4, 6, 7, 5)$**.** Let us return to your surveying task: you come upon a field with side-lengths 4, 6, 7, and 5. After measuring these side-lengths, you apply the Surveyor's Formula to produce an area estimate of $\frac{1}{2}(4+7) \cdot \frac{1}{2}(6+5) = 30.25$. Unbeknownst to you, the average area for this set of side-lengths (taken over the full range of possible θ values) is approximately 24.906 and the average relative error is approximately 24.498%.[2] However, you may not notice the error in the Surveyor's

[2]For consistency, all calculations will be displayed at five significant figures.

Formula if the value of θ for your particular field is not extremely large or small—see Figure 4.

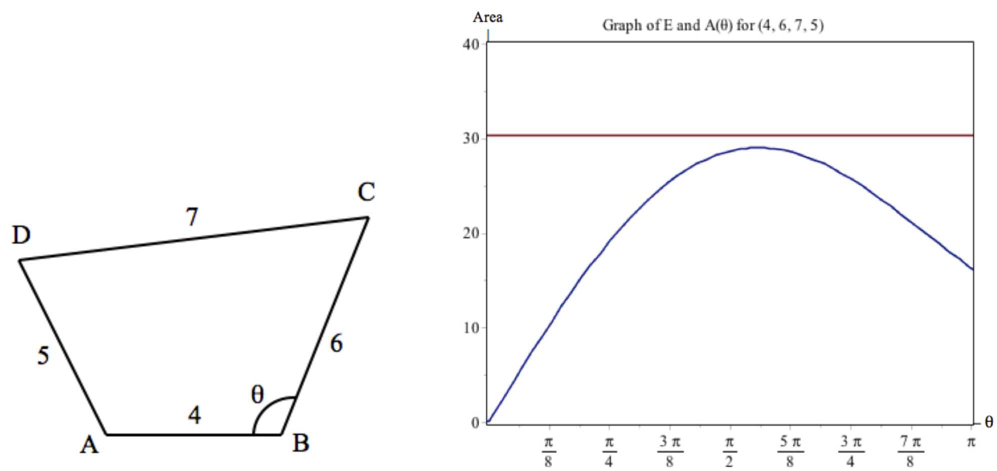

Figure 4. The quadrilateral $(4, 6, 7, 5)$, and a graph comparing $A(\theta)$ and E. Very large or small values of θ will produce highly erroneous estimates.

To apprehend this phenomenon more clearly, consider instead some more median intervals for θ. The easiest way to do this is take the maximum-area angle θ_0 and then use intervals centered on this value. In this example, θ_0 is approximately $100.75°$ (≈ 1.7583 radians) and the relative errors are:

Deviation from θ_0	Avg. Rel. Error
$\pm 2°$	4.3875%
$\pm 4°$	4.4327%
$\pm 6°$	4.5082%
$\pm 8°$	4.6140%
$\pm 10°$	4.7506%

While it is possible that you may notice a 5% relative error in your calculation, it is also possible that this error would pass unnoticed. One may also guess that the relative proportions of the four sides have a bearing on the relative error. Indeed, this seems to be the case, as the next example demonstrates.

Example 2: $(15, 3.5, 16, 4)$. This is a more "evenly balanced" quadrilateral than the previous one, in that opposite pairs of sides are more nearly equal. As a side note, this quadrilateral matches the side-lengths of a field described on the Temple of Horus.[3]

Here, your calculation gives $\frac{1}{2}(15+16) \cdot \frac{1}{2}(3.5+4) = 58.125$ for the area. The average area (again taken over all possible θ values) is 45.060, while the average relative error is 39.131%.

[3]See [5], [2] for more detail on this fact.

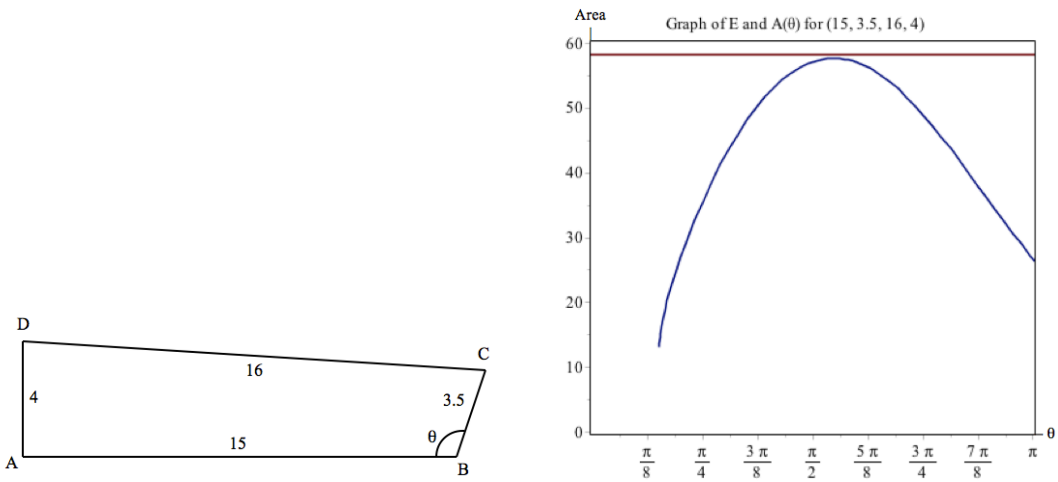

Figure 5. The quadrilateral $(15, 3.5, 16, 4)$, and the graphs of $A(\theta)$ and E.

In this case $\theta_0 \approx 98.577°$ (about 1.7205 radians), and the relative errors are:

Deviation from θ_0	Avg. Rel. Error
$\pm 2°$	0.93124%
$\pm 4°$	0.98289%
$\pm 6°$	1.0691%
$\pm 8°$	1.1902%
$\pm 10°$	1.3464%

The formula is more accurate (by a factor of four) than in the last example. It is far less likely that you would notice anything amiss with the Surveyor's Formula in this case.

5 Measures of Unevenness

Of course, there are several issues with this analysis. One is that you and your fellow surveyors do not have access to a wide range of farm fields to analyze, and most of the fields you encounter are roughly rectangular by design. Another is that the concept of "unevenness" is a nebulous one, and many different definitions could be used. To resolve this second issue, we choose to measure unevenness in the following way.

Definition 2. *Given a quadrilateral $ABCD$ with the standard orientation, define the unevenness measure μ as the difference between θ_0 (measured in radians) and $\frac{\pi}{2}$:*

$$\mu = \left|\theta_0 - \frac{\pi}{2}\right|.$$

Since $0 \leq \theta_0 \leq \pi$, it follows that $0 \leq \mu \leq \frac{\pi}{2}$.

In this way, a quadrilateral with $\mu = 0$ is one for which the "ideal" angle is a right angle. Furthermore, since $\theta = \theta_0$ implies that θ and φ are supplementary, a quadrilateral with $\mu = 0$ is one that can be decomposed into two right triangles by slicing along the diagonal AC.

Next we add two quadrilaterals to the list: $(22, 4, 23, 4)$[4] and $(10, 4.5, 10.5, 4)$. Let us now compare unevenness to average relative error for all four quadrilatrals:

Sides:	$(4, 6, 7, 5)$	$(15, 3.5, 16, 4)$	$(22, 4, 23, 4)$	$(10, 4.5, 10.5, 4)$
μ:	0.18754	0.14970	0.12533	0.034490
Deviation	Average Relative Error			
$\pm 2°$	4.3875%	0.93124%	0.81042%	0.22511%
$\pm 4°$	4.4327%	0.98289%	0.87018%	0.29071%
$\pm 6°$	4.5082%	1.0691%	0.97002%	0.40034%
$\pm 8°$	4.6140%	1.1902%	1.1103%	0.55448%
$\pm 10°$	4.7506%	1.3464%	1.2916%	0.75377%

It is easy to see that a decrease in μ corresponds to a decrease in average relative error.

Next, we repeat this analysis for a larger set of quadrilaterals. Specifically, 100 quadrilaterals were generated randomly[5] and the average relative errors were computed for an interval deviating from θ_0 by $\pm 6°$. When these errors are plotted against μ, the chart in Figure 6 is obtained.

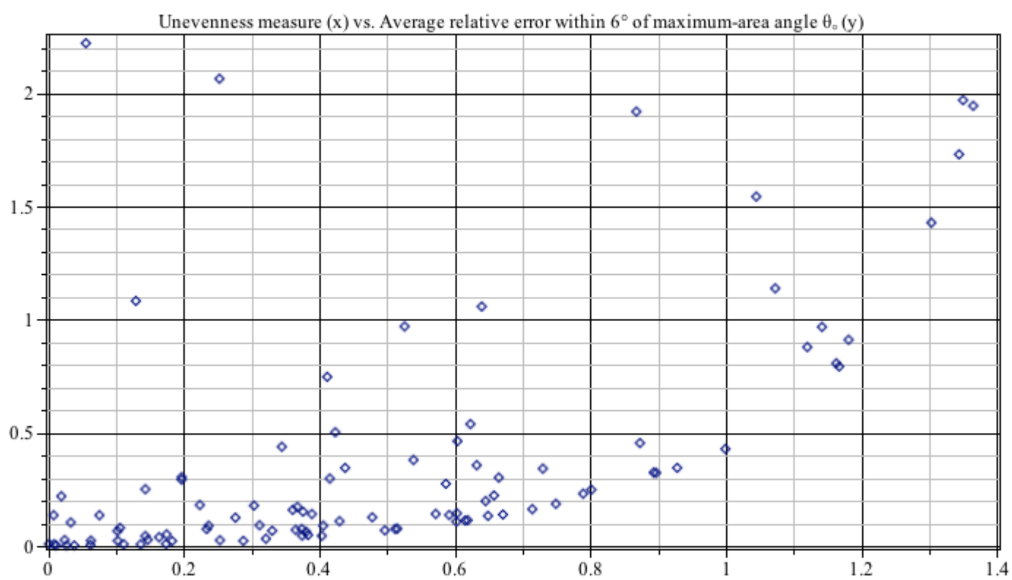

Figure 6.

[4]This quadrilateral also appears on the Temple of Horus in Edfu.

[5]All sides were restricted an to integer length between 1 and 40, and the fourth side-length was chosen with the restriction that it be smaller than the sum of the previous three.

Here, the trend is much clearer. It is interesting to note the presence of three outliers in the upper left of the chart; these are the quadrilaterals $(1, 27, 27, 4)$, $(2, 19, 20, 2)$, and $(1, 25, 23, 7)$. In each of these cases, the two longest sides are adjacent to each other, and greatly exceed the lengths of the two smaller sides (i.e., these are kite-shaped quadrilaterals). The upper-center outlier is $(5, 29, 38, 7)$.

Lastly, when $\mu = 0$, finding the difference between the Surveyor's Formula and the actual area is a matter of halving the difference of pairs of opposite sides (called the *difference average* to distinguish it from the common notion of an average).

Theorem 3. *Let $ABCD$ be a quadrilateral with the standard orientation. The estimated area given by the Surveyor's Formula exceeds the actual area by at least the product of the difference averages of opposite pairs of sides, i.e.,*

$$E - A(\theta_0) \;\geq\; \frac{1}{2}|a-c| \cdot \frac{1}{2}|b-d|.$$

Proof. First, from formula (2) we know that slicing the quadrilateral along the diagonal AC produces two triangles, from which it follows that Area $= \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sin\varphi \leq \frac{1}{2}ab + \frac{1}{2}cd$. Alternatively, if we slice the quadrilateral along BD we have Area $\leq \frac{1}{2}ad + \frac{1}{2}bc$.

Next, note that it suffices to show that $A(\theta_0) + \frac{1}{2}|a-c| \cdot \frac{1}{2}|b-d| \leq \frac{1}{2}(a+c) \cdot \frac{1}{2}(b+d)$. Recalling that the standard orientation dictates that $c \geq a$, we can see that

$$\begin{aligned} A(\theta_0) + \frac{1}{2}|a-c| \cdot \frac{1}{2}|b-d| &\leq \frac{1}{2}ab + \frac{1}{2}cd + \frac{1}{4}(c-a)(b-d) \\ &= \frac{1}{2}(a+c) \cdot \frac{1}{2}(b+d), \end{aligned}$$

provided $b \geq d$. In the case $d \geq b$, we merely use the other inequality from above:

$$\begin{aligned} A(\theta_0) + \frac{1}{2}|a-c| \cdot \frac{1}{2}|b-d| &\leq \frac{1}{2}ad + \frac{1}{2}bc + \frac{1}{4}(c-a)(d-b) \\ &= \frac{1}{2}(a+c) \cdot \frac{1}{2}(b+d). \quad \square \end{aligned}$$

6 Conclusion

The Surveyor's Formula, $\frac{1}{2}(a+c) \cdot \frac{1}{2}(b+d)$, was in widespread use in many cultures around the world over the course of many centuries. It is conceivable that a few perceptive individuals recognized that the formula was incorrect in some cases, and that it was only applicable when the quadrilateral in question was roughly rectangular. However, no evidence of this has been found (or is likely to be found). Nevertheless, the current analysis has led to the conclusion that errors in calculation are indeed small enough to not have been noticed, provided that the unevenness of the quadrilateral is sufficiently small.

Furthermore, Theorem 3 suggests that an improvement can be made by subtracting the product of the difference averages from the estimate given by the Surveyor's Formula:

$$\frac{1}{2}(a+c)\cdot\frac{1}{2}(b+d)-\frac{1}{2}|a-c|\cdot\frac{1}{2}|b-d|. \tag{7}$$

Theorem 3 assures us that this will always be greater than or equal to the actual area. Returning once again to your task as an ancient Egyptian surveyor, this new formula would not unnecessarily complicate your work: after measuring the four sides, it would only be necessary to subtract an additional term when doing the calculation. Furthermore, this formula can be reduced to $\frac{1}{2}(ab+cd)$ when $b \geq d$, or to $\frac{1}{2}(ad+bc)$ when $d \geq b$.[6] This version of the formula only requires you to multiply the lengths of two adjacent sides and average the resulting numbers. Since $a \leq c$ under the standard orientation, the reduced version of formula (7) can be summarized as follows:

> *Improved Surveyor's Formula*: Given the two pairs of opposite sides, separately multiply the smaller from each pair by the larger from the other pair. Then take the average of these two products.

This formula still employs the averaging technique of the Surveyor's Formula, but relies instead on the average of two products (as opposed to the product of two averages). Moreover, it does not rely on the particular orientation of the quadrilateral. The relative error data for the same 100 quadrilaterals makes it clear that formula (7) really is an improvement—see Figure 7.

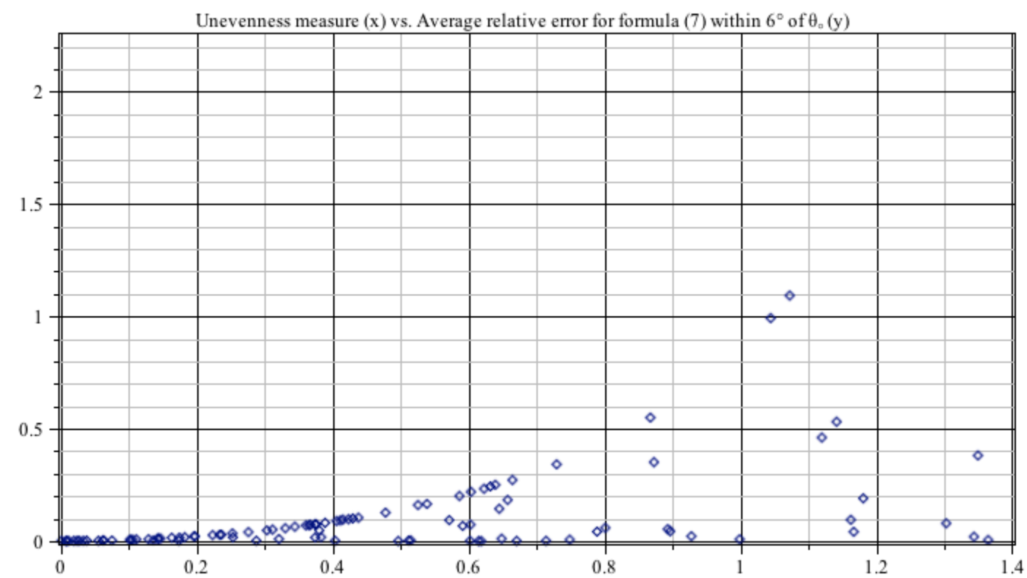

Figure 7.

[6]Interestingly, each of these formulas appears in Gupta [1, p. 55], though with a different purpose in mind.

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