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Eric Iksoon Im
University of Hawaii at Hilo

David L. Hammes
University of Hawaii at Hilo

Doug Wills
University of Washington - Tacoma Campus, dtwills@uw.edu

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STATIONARITY CONDITION FOR AR INDEX PROCESS

Eric Iksoon Im and David L. Hammes
University of Hawaii at Hilo

Douglas T. Wills
University of Washington at Tacoma

The stationarity conditions for an autoregressive (AR) process in general are reduced to a remarkably simple inequality if the lag coefficients are restricted to be identical. The condition is not only analytically elegant but also applicable in checking the validity of the stationarity conditions for such a restricted AR process of any order.

1. PROBLEM AND MOTIVATION

The stationarity conditions for an unrestricted autoregressive (AR) process in terms of lag coefficients increase in number with the lag order. This makes it progressively more complex to identify exact bounds for lag coefficients ensuring the stationarity. To reduce the level of complexity, we consider a restricted AR process where the lag coefficients are all equal (referred to in this note as an AR index process). Then, the AR index process of order n (≥ 1) can be written as

$$y_t + \alpha \sum_{i=1}^{n} y_{t-i} = \varepsilon_t, \tag{1}$$

where $\varepsilon_t \sim iid(0, \sigma^2)$ with $\sigma^2 < \infty$.

Because ε_t is stationary, the stationarity of y_t in (1) is contingent upon the convergence of y_t in the corresponding homogeneous difference equation:

$$y_t + \alpha \sum_{i=1}^{n} y_{t-i} = 0. \tag{2}$$

In light of the Schur theorem (Chiang and Wainwright, 2005, pp. 589–590), the necessary and sufficient conditions for convergence of y_t in (2) can be expressed as

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$$|\Sigma_{2k}(\alpha)| > 0 \quad \text{for all } k\ (1 \le k \le n), \tag{3}$$

where

$$\Sigma_{2k}(\alpha) = \begin{bmatrix} S_k(\alpha) & T_k'(\alpha) \\ T_k(\alpha) & S_k'(\alpha) \end{bmatrix}, \tag{4}$$

in which $S_k(\alpha)$ defines a lower triangular matrix of order k whose (i, j)th element is equal to α if $i > j$, 1 if $i = j$, and 0 if $i < j$, whereas $T_k(\alpha)$ is also a lower triangular matrix of order k whose (i, j)th element is equal to α if $i \ge j$ and 0 if $i < j$. That is,

$$\Sigma_{2k}(\alpha) = \begin{bmatrix} 1 & 0 & \dots & 0 & \alpha & \dots & \dots & \alpha \\ \alpha & \ddots & \ddots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ \alpha & \dots & \alpha & 1 & 0 & \dots & 0 & \alpha \\ & & & & & & & \\ \alpha & 0 & \dots & 0 & 1 & \alpha & \dots & \alpha \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \alpha \\ \alpha & \dots & \dots & \alpha & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{5}$$

In this simplified framework, the set of convergence conditions in (3) reduces to a single inequality in terms of α and n only (established formally as a theorem in the next section):

$$-1/n < \alpha < 1. \tag{6}$$

If we let $\alpha = -\phi$ in (3) as is usual in the econometrics literature, the convergence condition (6) is expressed in terms of ϕ as

$$-1 < \phi < 1/n. \tag{7}$$

In addition to its analytical elegance, either (6) or (7), depending on how the AR index process is written, is applicable in practice. The stationarity condition for an AR(1) process, $y_t = \phi y_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim iid(0, \sigma^2)$, known as $-1 < \phi < 1$, is immediately obtained from (7) for $n = 1$. The stationarity conditions for an AR(2) process, $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t$, are also known as $\phi_1 + \phi_2 < 1$; $\phi_2 - \phi_1 < 1$; $-1 < \phi_2 < 1$ (e.g., see Judge, Griffiths, Hill, Lütkepohl, and Lee, 1985, p. 293). The solution of these three inequalities under the restriction $\phi_1 = \phi_2 = \phi$ can be worked out as $-1 < \phi < \frac{1}{2}$, but the same result is immediately derived from (7) for $n = 2$. Similarly, the condition can be used to check the validity of the stationarity conditions for an AR index process of any higher order.

2. PROOF AND DISCUSSION

The convergence condition in (6) depends on analytical eigenvalues of $\Sigma_{2k}(\alpha)$ that may well be useful elsewhere. Hence, we deal with them separately in a pair of lemmas in sequence prior to establishing our main result as a theorem.

DEFINITION. *$\lambda_i(\cdot)$ denote eigenvalues of any real square matrix of order r in descending order: $\lambda_1(\cdot) \geq \lambda_2(\cdot) \geq \cdots \geq \lambda_r(\cdot)$.*

LEMMA 1. *For any real square matrix M of order r, $\lambda_i(\xi_1 M + \xi_2 I_r) = \xi_1 \lambda_i(M) + \xi_2$ where ξ_1 and ξ_2 are nonnull constants.*

Proof of Lemma 1. Because $|\lambda I_r - (\xi_1 M + \xi_2 I_r)| = |(\lambda - \xi_2) I_r - \xi_1 M|$, we can deduce $\lambda = \lambda_i(\xi_1 M + \xi_2 I_r)$ and $\lambda - \xi_2 = \lambda_i(\xi_1 M) = \xi_1 \lambda_i(M)$. Substituting out λ from either equation, $\lambda_i(\xi_1 M + \xi_2 I_r) = \xi_1 \lambda_i(M) + \xi_2$.[1] ■

LEMMA 2. *For $\Sigma_{2k}(\alpha)$ defined in (5), $\lambda_i(\Sigma_{2k}(\alpha)) = k\alpha$ for $i = 1$; 1 for $i = 2,3,\ldots, k-1$; $1 - \alpha$ for $i = k+1, k+2, \ldots, 2k-1, 2k$.*

Proof of Lemma 2. Define a square matrix of order $2k$:

$$A_{2k} = \Sigma_{2k}(1) = \begin{bmatrix} L_k & L'_k \\ L_k & L'_k \end{bmatrix}, \tag{8}$$

where L_k is a lower triangular matrix of order k whose (i,j)th element is equal to 1 if $i \geq j$ and 0 if $i < j$.

Then we can rewrite $\Sigma_{2k}(\alpha)$ in terms of A_{2k} as

$$\Sigma_{2k}(\alpha) = \alpha A_{2k} + (1-\alpha) I_{2k}. \tag{9}$$

Hence, in view of Lemma 1

$$\begin{aligned} \lambda_i(\Sigma_{2k}(\alpha)) &= \lambda_i(\alpha A_{2k} + (1-\alpha) I_{2k}) \\ &= \alpha \lambda_i(A_{2k}) + (1-\alpha). \end{aligned} \tag{10}$$

It is clear from (10) that once analytical solutions for $\lambda_i(A_{2k})$ are obtained, their counterparts for $\lambda_i(\Sigma_{2k}(\alpha))$ immediately follow.

The upper k rows in A_{2k} in (8) are identical with the lower k rows, which implies $rank(A_{2k}) = k$. Hence, we can reduce the lower k rows to null vectors through pre- and postmultiplications of an appropriate nonsingular matrix and its inverse:

$$P_{2k}^{-1} A_{2k} P_{2k} = B_{2k} = \begin{bmatrix} L_k + L_k' & L_k' \\ 0_{k\times k} & 0_{k\times k} \end{bmatrix}, \tag{11}$$

where

$$P_{2k} = \begin{bmatrix} I_k & 0_{k\times k} \\ I_k & I_k \end{bmatrix}; \qquad P_{2k}^{-1} = \begin{bmatrix} I_k & 0_{k\times k} \\ -I_k & I_k \end{bmatrix}. \tag{12}$$

Because P_{2k} in (12) is nonsingular, A_{2k} and B_{2k} in (11) have the same set of eigenvalues with the same multiplicities (Magnus and Neudecker, 1990, p. 14). Hence, noting that B_{2k} is block-diagonal,

$$\lambda_i(A_{2k}) = \lambda_i(B_{2k}) = \begin{cases} \lambda_i(L_k + L_k') & \text{for } i = 1,2,\ldots,k; \\ 0 & \text{for } i = k+1, k+2,\ldots,2k. \end{cases} \tag{13}$$

Because $L_k + L_k' = \iota\iota' + I_k$ where $\iota(k \times 1) = (1, 1,\ldots,1)'$, in view of Lemma 1,

$$\begin{aligned} \lambda_i(L_k + L_k') &= \lambda_i(\iota\iota' + I_k) \\ &= \lambda_i(\iota\iota') + 1, \end{aligned} \tag{14}$$

in which

$$\lambda_i(\iota\iota') = \begin{cases} k & \text{for } i = 1; \\ 0 & \text{for } i = 2,3,\ldots,k. \end{cases} \tag{15}$$

Recursive substitutions of (15) into (14), (14) into (13), and then (13) into (10) lead to the eigenvalues sought after:

$$\lambda_i(\Sigma_{2k}(\alpha)) = \begin{cases} k\alpha + 1 & \text{for } i = 1; \\ 1 & \text{for } i = 2,3,\ldots,k; \\ 1 - \alpha & \text{for } i = k+1, k+2,\ldots,2k-1,2k. \end{cases} \qquad \blacksquare$$

THEOREM. *For $y_t + \alpha \sum_{i=1}^{n} y_{t-i} = 0$, the necessary and sufficient condition for convergence of y_t is $-1/n < \alpha < 1$.*

Proof of Theorem. y_t in the theorem converges if and only if $|\Sigma_{2k}(\alpha)| > 0$ for all k $(1 \le k \le n)$ by virtue of the Schur theorem as stated in (3). Reflecting the eigenvalues from Lemma 2 on (3),

$$|\Sigma_{2k}(\alpha)| = \prod_{i=1}^{2k} \lambda_i(\Sigma_{2k}(\alpha)) = (k\alpha + 1)^1 (1)^{k-1} (1-\alpha)^k$$
$$= (k\alpha + 1)(1-\alpha)^k > 0 \qquad \text{for all } k\ (1 \le k \le n), \tag{16}$$

which holds if and only if $-1/n < \alpha < 1$. ■

NOTE

1. We owe conciseness of the proof to an anonymous referee.

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